Mathematics 290-A **Exam 2** **Fall 2006**

October 3, 2006 **Mister Key**

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Name

Technology used: ______________________

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**
- **When given a choice, be sure to specify which problem(s) you want graded.**

Do any three (3) of these computational problems

C.1. Do all of the following.

(a) Show that the set of vectors $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

i. $A = \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 2 & 1 \end{bmatrix}$ has row echelon form: $B = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, Thus the homogenous linear system $x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a free variable and so there are nontrivial solutions. Any such solution (like $x_1 = 5$, $x_2 = 0$, $x_3 = -3$, $x_4 = 1$) gives a nontrivial relation of linear dependence for the vectors in S making S linearly dependent

(b) Find two vectors $\vec{w}_1, \vec{w}_2$ that are both in S and for which $< S >=< T >$, where $T = \{\vec{w}_1, \vec{w}_2\}$.

i. We can't find two vectors whose span equals the span of S but we can find three. By throwing out the last vector in S (because it is associated with a free variable), we get $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and Theorem BS tells us that $< S >=< T >$.

(c) Write one of the extra vectors in S as a linear combination of $\vec{w}_1$, and $\vec{w}_2$.

i. Using our solutions from part (a) we see $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

C.2. Write all of the following complex numbers in the form $a + bi$.

(a) $2(2 - 3i) - 7(6 + 2i) = -38 - 20i$

(b) $\frac{4+3i}{2-i} = \frac{4+3i}{2-i}\frac{2+1}{2+1} = \frac{10+5i}{5} = 2 + i$

(c) $\sqrt{i}$ [Hint: write $(a + bi)^2 = i$ and solve a system of equations.]

i. $(a + bi)^2 = i$ gives $a^2 - b^2 + 2abi = 0 + i$

ii. so $a^2 - b^2 = 0$, and $2ab = 1$.

iii. $a = \pm b$ and substituting gives $\pm 2b^2 = 1$. Using the plus sign we have $b = \frac{1}{\sqrt{2}}$ and choosing $a = b = \frac{1}{\sqrt{2}}$ we see that one square root of i is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

C.3. The vectors $\vec{u}_1, \vec{u}_2$,and $\vec{u}_3$ below are already orthonormal. Use the Gram-Schmidt procedure to find a vector $\vec{u}_4$ so that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an **orthonormal** set.

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find all vectors $\overrightarrow{v}_4$ in R^4 so that $\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3, \overrightarrow{v}_4$ form an orthonormal set.

(a) The Gram-Schmidt formula is

$$\vec{u}_i = \vec{v}_i - \left(\frac{<\vec{v}_i, \vec{u}_1>}{<\vec{u}_1, \vec{u}_1>}\right)\vec{u}_1 - \cdots - \left(\frac{<\vec{v}_i, \vec{u}_{i-1}>}{<\vec{u}_{i-1}, \vec{u}_{i-1}>}\right)\vec{u}_{i-1}$$

so

$$\begin{aligned} \vec{u}_4 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{<\vec{v}_{4i}, \vec{u}_1>}{<\vec{u}_1, \vec{u}_1>}\right)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \left(\frac{<\vec{v}_{4i}, \vec{u}_{12}>}{<\vec{u}_2, \vec{u}_2>}\right)\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} - \left(\frac{<\vec{v}_{4i}, \vec{u}_{32}>}{<\vec{u}_3, \vec{u}_3>}\right)\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1/2}{1}\right)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \left(\frac{1/2}{1}\right)\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} - \left(\frac{1/2}{1}\right)\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \end{aligned}$$

(b) To guarantee the vectors are **orthonormal**, we divide by $<\vec{u}_4, \vec{u}_4> = \sqrt{4/16} = \frac{1}{2}$ giving a new $\vec{u}_4 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

C.4. Compute the following matrix-vector product **by hand** in two ways.

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Using term by term multiplication: $\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}. = \begin{bmatrix} 5+2+3 \\ -20+2+3 \\ 10-6+15 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \\ 19 \end{bmatrix}$

(b) Using the definition: $\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \\ 19 \end{bmatrix}$

Do any two (2) of these problems from the text, homework, or class.

You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}$ is a linearly independent set and that $\mathbf{v} \notin < S >$. Prove the set $W = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{v}\}$ is a linearly independent set.

(a) Using the definition. Let

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_p \mathbf{u}_p + \alpha_{p+1} \mathbf{v} = \mathbf{0} \tag{1.}$$

be a relation of linear dependence. We show that the only way this equation can be true is if all of the α's equal 0.

(b) If $\alpha_{p+1} \neq 0$ then we can write $\mathbf{v}$ as a linear combination of the other vectors $\mathbf{v} = \frac{-\alpha_1}{\alpha_{p+1}} \mathbf{u}_1 - \cdots - \frac{-\alpha_p}{\alpha_{p+1}} \mathbf{u}_p$. But we know $\mathbf{v}$ is not in the span of S so this is impossible. Hence we can conclude that a_{p+1} **must** be zero in equation (1.) and so that equation can be rewritten as

$$\begin{aligned} \mathbf{0} &= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_p \mathbf{u}_p + (0)\,\mathbf{v} \\ &= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_p \mathbf{u}_p \end{aligned}$$

(c) Now the linear independence of S tells us the rest of the α's are also 0 and we are done.

M.2. Suppose $S = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a linearly independent set in R^5. Is the set of vectors $2\overrightarrow{v}_1 + \overrightarrow{v}_2 + 3\overrightarrow{v}_3,\ \overrightarrow{v}_2 + 5\overrightarrow{v}_3,\ 3\overrightarrow{v}_1 + \overrightarrow{v}_2 + 2\overrightarrow{v}_3$ linearly dependent or independent?

(a) Using the definition we consider an arbitrary relation of linear dependence and then re-write it by collecting on $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$$\begin{aligned} \vec{0} &= a\,(2\overrightarrow{v}_1 + \overrightarrow{v}_2 + 3\overrightarrow{v}_3) + b\,(\overrightarrow{v}_2 + 5\overrightarrow{v}_3) + c\,(3\overrightarrow{v}_1 + \overrightarrow{v}_2 + 2\overrightarrow{v}_3) \\ &= (2a + 0b + 3c)\,\vec{v}_1 + (a + b + c)\,\vec{v}_2 + (3a + 5b + 2c)\,\vec{v}_3 \end{aligned}$$

(b) Since S is linearly independent we know each of the coefficients above must equal zero giving rise to the homogeneous system of linear equations with augmented matrix $\left[A|\vec{0}\right] = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 5 & 2 & 0 \end{bmatrix}$.

Row reducing we obtain $B = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus we see that the linear system has infinitely many solutions – one of which is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.This means S is linearly dependent since there are many non-trivial relations of linear dependence, one of which is

$$\vec{0} = -3\,(2\overrightarrow{v}_1 + \overrightarrow{v}_2 + 3\overrightarrow{v}_3) + (1)\,(\overrightarrow{v}_2 + 5\overrightarrow{v}_3) + 2\,(3\overrightarrow{v}_1 + \overrightarrow{v}_2 + 2\overrightarrow{v}_3)$$

M.3. Prove Theorem TMA, Transpose and Matrix Addition.

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

(a) This proof is in the textbook on page 206.

Do one (1) of these problems you've not seen before.

T.1. Suppose A is a square matrix of size n satisfying $A^2 = AA = O$. Prove that the only vector $\vec{x}$ satisfying $(I_n - A)\,\vec{x} = \vec{0}$ is the zero vector.

(a) We algebraically manipulate the equation

$$\begin{aligned} (I_n - A)\,\vec{x} &= \vec{0} \\ A\,(I_n - A)\,\vec{x} &= A\vec{0} \\ \left(AI_n - A^2\right)\vec{x} &= \vec{0} \\ A\vec{x} - A^2\vec{x} &= \vec{0} \\ A\vec{x} - O\vec{x} &= \vec{0} \\ A\vec{x} - \vec{0} &= \vec{0} \\ A\vec{x} &= \vec{0} \end{aligned}$$

T.2. Recall that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Now explain why the fact that $\begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -4 & -2 & -2 & 0 & 1 & 0 \\ -5 & -2 & -4 & 0 & 0 & 1 \end{bmatrix}$ has reduced row-echelon form $\begin{bmatrix} 1 & 0 & 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 0 & -\frac{5}{2} & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$ tells us the only vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ that can be in the span of $S = \left\{ \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix} \right\}$ are those where $a + 2b - c = 0$.

(a) We know that $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the span of S if and only if the following system of linear equations is consistent

$$\begin{aligned} 3x + 2y + 0z &= a \\ -4x - 2y - 2z &= b \\ -5x - 2y - 4z &= c \end{aligned}$$

(b) But we can write this as

$$\begin{aligned} 3x + 2y + 0z &= (1)\,a + (0)\,b + (0)\,c \\ -4x - 2y - 2z &= (0)\,a + (1)\,b + (0)\,c \\ -5x - 2y - 4z &= (0)\,a + (0)\,b + (1)\,c \end{aligned}$$

(c) Now note that running elementary operations on the matrix B below uses the last three columns to

$$B = \begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -4 & -2 & -2 & 0 & 1 & 0 \\ -5 & -2 & -4 & 0 & 0 & 1 \end{bmatrix}$$

keep track of how many a, b, and c's there are if we were to run those same elementary row operations by hand on the augmented matrix A.

$$A = \begin{bmatrix} 3 & 2 & 0 & a \\ -4 & -2 & -2 & b \\ -5 & -2 & -4 & c \end{bmatrix}$$

Thus the reduced echelon form of B tells us that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 2 & (0)\,a + b - c \\ 0 & 1 & -3 & (0)\,a - \frac{5}{2}b + 2c \\ 0 & 0 & 0 & a + 2b - c \end{bmatrix}$$

which represents a consistent system if and only if $a + 2b - c$ is not 0.